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$\lambda\phi^4$ q-Renormalization Program

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A regularization scheme for quantum field theories given in a q-mutator algebra for the internal momentum space in a loop integration is constructed. We show Feynman integrals that are finite for $q \neq 1$ but diverse as $q \to 1$. Using this regularization scheme, we propose a renormalization program in q-mutator space (q-renormalization program) for the $\lambda\phi^4$ theory as an example, up to some one-loop diagrams. This work paves the way to obtaining physically measurable quantities from quantum field theories over spaces that neither commute nor anticommute.

1. INTRODUCTION

It is well known that for any quantum field theory one can construct the Feynman rules for calculating the Green's functions and S-matrix elements in perturbation theory, but for relativistic field theory one often finds infinities in the calculation of diagrams containing loops. These divergences will render the calculation meaningless.

Renormalization is a prescription that allows us consistently to isolate and remove all these infinities from the physically measurable quantities. However, we should remark that the need for renormalization is rather general and is not unique to the relativistic field theories.

In any renormalization program one first introduces some appropriate regularization scheme so that all divergent integrals are made finite. Then we are free to manipulate formally these quantities, which are divergent only if the regularization is removed.

On the other hand, there has been a good deal of interest in recent years in the construction of quantum group structures in integrable confor-

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mal field theories and Chern–Simons holonomies (Smit, 1990; Guadagnini *et al.*, 1990). In addition, the reexamination of the question, "Can the Pauli principle be violated by a tiny amount?," has led to experimental reviews and devices to report the small violation of the Pauli principle (Greenberg and Mohapatra, 1989*a*; Ramberg and Snow, 1990); theoretical formulations for models which have a small violation of the exclusion principle (Greenberg and Mohapatra, 1987, 1989*b*; Greenberg, 1990*a*) include infinite statistics constructed using q-mutator algebras that define a quantum group.

Independently, some authors have been working in quantum field theories constructed over spaces that neither commute nor anticommute. In this way a new noncommutative geometric structure for quantum field theories (*e.g.*, Yang–Mills model) developed in the generalized quantum context has been constructed (Connes, 1988, 1990*a*,*b*), among other things, to better explain de Broglie's symmetry (Bacry, 1990).

Interesting reviews about the current generalized soliton theory show how such different subjects as Hopf algebras, quantum groups, knots, Jones polynomials, spin chains, Bose–Fermi equivalence, conformal field theory, quantum deformations, integrable lattice models, affine (Kac–Moody) Lie algebras, Virasoro algebras, strings, SUSY, and the Ising model are connected (Bullough and Timonen, 1990).

Following the previous ideas, we think that a regularization scheme and a renormalization program based on a q-mutator space can be used to obtain physically measurable quantities from quantum field theories on a noncommutative differential geometry.

In this paper we show a regularization scheme for any quantum field theory, applied to some one-loop graphs in $\lambda\phi^4$ as an example (Section 2), and a renormalization program for these particular cases (Section 3). Our conclusions are given in Section 4.

2. A q-REGULARIZATION SCHEME

In this section we introduce a regularization scheme constructed in noncommutative geometry.

Following an idea related to dimensional regularization, we assume a Haar weight as invariant integration over the Hopf *-algebra of the generalized noncommutative internal momentum space. This deformation is parameterized by q and reduces to the unregulated theory as $q \to 1$.

Whereas dimensional regularization involves the extension of the internal momentum space in the loop integration to n components with $\mathrm{Re}(n) < 4$, q-regularization assumes q-mutator structure for the internal momentum space in the loop integration. For the first case the finite (and

n-dependent) Feynman integrals are made divergent when $n \to 4$; in the second case the Haar weights (deformed versions of Feynman integrals) are finite (and q dependent) at $q \neq 1$; by contrast, as $q \to 1$, divergences come out.

We should remark that q is any element of a field and if $q \to 1$ we recover the usual differential geometry for the quantum field theory. Besides, it is well known (Fredenhagen, 1981) that for all cases where $q \neq 1$ locality is lost, then a relativistic quantum field theory cannot be constructed, but only a quantum mechanical one. For the case $-1 < q < 1$, even if locality has been lost, the TCP theorem follows (Greenberg, 1990*b*).

We do not concern ourselves with the lack of locality for the q-mutator momentum structure of the internal momentum space in the loop integration because it has nothing to do with the external momentum space (which is local) or any other measurable physical quantity.

To compute the Haar weight, we should choose a basis of C_q^n. As an algebra of the internal momentum space we define C_q^n as the complex algebra generated by 1 and the n generators l_m, $l \leq m \leq n$, with relations

$$[l_k, l_j] = il_j \sqrt{(1-q)} \qquad \begin{cases} k \text{ is an even number} \\ j \text{ is an odd number} \\ n = k + j \\ i = \sqrt{-1} \end{cases} \tag{2.1}$$

such that as $q \to 1$ the algebra becomes the commutative algebra of functions on C^n generated by l_m. We transform this into a *-algebra via

$$l_k^* = l_k, \qquad l_j^* = l_j q^{i/2} \qquad \begin{cases} k \text{ is an even number} \\ j \text{ is an odd number} \\ i = \sqrt{-1} \end{cases} \tag{2.2}$$

For every finite-dimensional Hopf C^* algebra (or Hopf–von Neumann algebra) there is an invariant integration, the Haar weight $\int$, unique up to a normalization.

We choose a basis of C_q^n of the form

$$B^{a_1 \ldots a_n} = e^{ia_1 l_1} \ldots e^{ia_n l_n}, \qquad a_m \in C \text{ (complex numbers)} \tag{2.3}$$

and the dual basis $D_{a'_1 \ldots a'_n} \in {}_q C^n$ defined via

$$B^{a_1 \cdots a_n} D_{a'_1 \ldots a'_n} = \delta(a'_1 - a_1) \ldots \delta(a'_n - a_n), \qquad a'_m \in C \tag{2.4}$$

where the δ functions have been defined with respect to the usual Lebesgue integration.

Then we propose, up to a normalization, the following:

$$\int B^{a_1 \ldots a_n} = \int da_1 \ldots da_n \, (B^{a_1 \cdots a_n} S^2 B^{a_1 \cdots a_n}) D_{a'_1 \ldots a'_n} \tag{2.5}$$

where S^2 is defined such that

$$S^2 l_k = l_k \qquad \text{and} \qquad S^2 l_j = q^{-1} l_j \qquad \begin{cases} k \text{ is an even number} \\ j \text{ is an odd number} \end{cases} \tag{2.6}$$

Then, from (2.1) and (2.4)–(2.6) it follows that

$$\int B^{a_1 \ldots a_n} = \int da_1 \ldots da_n \prod_i \delta(a'_k - (a'_k + a_k))$$
$$\times \prod_j \delta(a'_j - (a'_j q^{-1} + a_j e^{-a'_i \sqrt{(1-q)}})) \tag{2.7}$$

where k is an even number and j is an odd one.

Now, we can calculate the integral of a general element f in the internal momentum space C_q^n if we assume

$$f =: f^{cl}(l_k, l_j) := \int da_1 \ldots da_n f'(a_1 \ldots a_n) B^{a_1 \ldots a_n} \tag{2.8}$$

where

$$f'(a_1 \ldots a_n) = (2\pi)^{-2} \int db_1 \ldots db_n f^{cl}(b_1 \ldots b_n) e^{-ib_1 a_1} \ldots e^{-ib_n a_n}$$

is the Fourier transform of the normal ordered form for the function of the generators (putting l_k to the left, where k is an even number).

Then

$$\int f = \int da'_1 \ldots da'_n \prod_k e^{a'_k \sqrt{(1-q)}} f'(a'_j (1 - q^{-i}) e^{a'_k \sqrt{(1-q)}}) \tag{2.9}$$

for all k an even number and j an odd one.

Changing the order of the integration, we have (up to normalization)

$$\int f = [2\pi\delta(0)]^k \int \prod_j da'_j f'(0, a'_j (1 - q^{-i})) \tag{2.10}$$

for all j an odd number.

As $q \to 1$ we obtain $[2\pi\delta(0)]$ (a normalization) times the ordinary

$$\int f = (2\pi)^{-n} \int db_1 \ldots db_n f^{cl}(b_1 \ldots b_n) \tag{2.11}$$

which diverges, on the contrary, at $q \neq 1$ and assuming proper analycity and decay of f' to allow rotation of the contour, j of the $2\pi\delta(0)$ factors are replaced by $(1 - q^{-1})^{-j}$ and an integral, which can be finite for suitable f.

Another basis for C_q^n and examples have been studied recently (Majid, 1990).

We shall use the simple $\lambda\phi^4$ theory as an example to illustrate this regularization scheme. The Lagrangian density is separated into free and interacting parts,

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 \tag{2.12}$$

$$\mathscr{L}_0 = \frac{1}{2}[(\partial_\mu \phi_0)^2 - \mu_0^2 \phi_0^2] \tag{2.13}$$

$$\mathscr{L}_1 = -\frac{\lambda_0}{4!}\phi_0^4 \tag{2.14}$$

The propagator and the vertex of this theory are displayed in Fig. 1.

We will concentrate on the one-particle-irreducible (1PI) diagrams; they are the Feynman diagrams that cannot be disconnected by cutting any one internal line. Therefore we define the 1PI Green's functions Γ, which have contributions coming from 1PI diagrams only.

We select only 1PI diagrams because any one-particle-reducible diagram can be decomposed into 1PI diagrams without further loop integration. If we know how to take care of the divergences of 1PI diagrams, we will also be able to handle reducible diagrams.

Since there is no divergence in the tree (zero-loop) diagrams, we illustrate our calculation with the one-loop divergent 1PI diagrams in the $\lambda\phi^4$ theory of Fig. 2.

Figure 2 shows the vertex corrections with contributions given by

$$\Gamma^a = \Gamma(s) = \frac{(-i\lambda_0)^2}{2}\int \frac{d^4 l}{(2\pi)^4}\frac{i}{(l-p)^2 - \mu_0^2 + i\varepsilon}\frac{i}{l^2 - \mu_0^2 + i\varepsilon} \tag{2.15}$$

$$\Gamma^b = \Gamma(t); \qquad \Gamma^c = \Gamma(u) \tag{2.16}$$

where

$$s = p^2 = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2 \tag{2.17}$$

are the Mandelstam variables. These corrections diverge logarithmically.

If we consider the (2.15) and (2.16) integrals as Haar weights over the (2.3) basis of C_q^4 (noncommutative internal momentum space) defined by

Fig. 1.

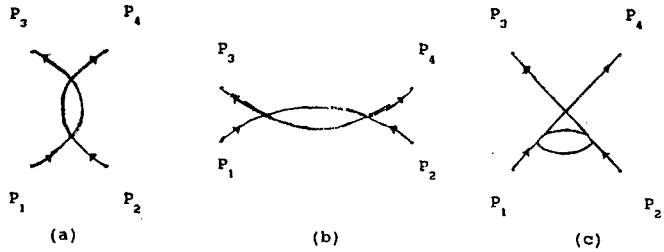

Fig. 2.

(2.10), we obtain

$$\Gamma_q^a = \Gamma_q(s) = \frac{\lambda_0^2 \delta(0)}{2(2\pi)^3} \times \int \frac{d^j l'_j}{\{p_k^2 + [l'_j(1-q^{-i}) - p_j]^2 - \mu_0^2 + i\varepsilon\}\{[l'_j(1-q^{-i})]^2 - \mu_0^2 + i\varepsilon\}} \tag{2.18}$$

$$\Gamma_q^b = \Gamma_q(t), \qquad \Gamma_q^c = \Gamma_q(u) \tag{2.19}$$

where s, t, and u are the Mandelstam variables, l'_j are the odd components of the dual internal momentum in noncommutative geometry, and $p_k(p_j)$ are the even (odd) components of the external momentum in the well-known commutative space-time.

Unless $q = 1$, (2.18) and (2.19) are finite; thus, we have a regularization scheme. We should remark that the limit $q \to 1$ applied to (2.18) and (2.19), quadratically divergent, does not lead straightforwardly to (2.15) and (2.16) because in this case our Haar weight formulation has no sense, which means that (2.1) reduces to the standard commutative case.

3. A q-RENORMALIZATION PROGRAM

To formulate the (2.18) and (2.19) Haar weights we have used the (2.3) basis for the noncommutative internal momentum space C_q^4. Let us assume another $B^{a_1 \cdots a_n} D_{a'_1 \ldots a'_n}$ basis (dual basis) set, such that we have general expressions for Γ_q^a, Γ_q^b, and Γ_q^c given by

$$\Gamma_q^a = \Gamma_q(s) = \frac{\lambda_0^2 \delta(0)}{2(2\pi)^3} \int \frac{d^j l'_j}{[a^2 - (bl'_j - c)^2 - d](b^2 l'_j - d)} \tag{3.1}$$

$$\Gamma_q^b = \Gamma_q(t) \qquad \text{and} \qquad \Gamma_q^c = \Gamma_q(u) \tag{3.2}$$

where s, t, and u are the Mandelstam variables and l'_j are the odd components of the dual internal momentum in noncommutative geometry.

Of course, (3.1) and (3.2) reduce to (2.18) and (2.19) in the case $a^2 = p_k^2$, $b = (1 - q^{-i})$, $c = p_j$, $d = \mu_0^2 - i\varepsilon$, where p_k (p_j) are the even (odd) components of the external momentum in the usual commutative space-time. We should remark that, for the general case, the set $\{a, b, c, d\}$ always includes some functionals of the Mandelstam variables.

Solving (3.1) and (3.2), we obtain the following results.

Case I. $e > c^2$, with $e = c^2 + a^2 - d$:

$$\Gamma_q^a = \Gamma_q(s) = \frac{-\lambda_0^2\delta(0)}{2(2\pi)^3 b[(d+e)^2 - 4c^2 d]}\left\{2c \ln b + \frac{2c^2 - d - e}{(e-c^2)^{1/2}}\frac{\pi}{2} + c\ln(-1)^z\frac{eb^2}{d} + \frac{2c^2-d-e}{(e-c^2)^{1/2}}\tan^{-1}\left[\frac{-c}{(e-c^2)^{1/2}}\right]\right\} \tag{3.3}$$

where

$$z = -[(d+e)/2\sqrt{d} + c] \tag{3.4}$$

and

$$\Gamma_q^b = \Gamma_q(t), \qquad \Gamma_q^c = \Gamma_q(u) \tag{3.5}$$

for the corresponding Mandelstam variables.

Case II. $e < c^2$, with $e = c^2 + a^2 - d$:

$$\Gamma_q^a = \Gamma_q(s) = \frac{-\lambda_0^2\delta(0)}{2(2\pi)^3 b[(d+e)^2 - 4c^2 d]}\left[2c\ln b + c\ln(-1)^z\frac{eb^2}{d} + \frac{2c^2-d-e}{(c^2-e)^{1/2}}\ln\frac{-bc - b(c^2-e)^{1/2}}{-bc + b(c^2-e)^{1/2}}\right] \tag{3.6}$$

It is possible to prove that, if in the limit $q \to 1$,

$$e \to 4\mu^2, \qquad c \to \begin{cases} (p_1 + p_2) = p_a & \text{for } \Gamma_q^a \\ (p_1 - p_3) = p_b & \text{for } \Gamma_q^b \\ (p_1 - p_4) = p_c & \text{for } \Gamma_q^c \end{cases} \tag{3.7}$$

and b, d are defined in such a way that we have the following results:

Case I. $e > c^2$:

$$\ln b = \frac{[\pi(4\mu_0^2 - p^2)^{1/2} - 2p](2p^2 - d - 4\mu_0^2)}{8p(4\mu_0^2 - p^2)} - \frac{1}{4}\ln(-1)^z\frac{4\mu_0^2}{d} \tag{3.8a}$$

$$d + \frac{8p(4\mu_0^2 - p^2)}{\pi(4\mu_0^2 - p^2)^{1/2} - 2p}\ln\frac{2p^2 - d - 4\mu_0^2}{(d + 4\mu_0^2)^2 - 4p^2 d} = (2p^2 - 4\mu_0^2) - \frac{8p(4\mu_0^2 - p^2)}{\pi(4\mu_0^2 - p^2)^{1/2} - 2p}\ln\frac{-\lambda_0^2\delta(0)(-1)^{z/4}p(4\mu_0^2)^{1/4}}{\pi(4\mu_0^2 - p^2)} \tag{3.8b}$$

Case II. $e < c^2$:

$$\ln b = \frac{-(2p^2 - d - 4\mu_0^2)(2 + i\pi)}{4(p^2 - 4\mu_0^2)} - \frac{1}{4} \ln(-1)^z \frac{4\mu_0^2}{d} \tag{3.9a}$$

$$d - \frac{4(p^2 - 4\mu_0^2)}{2 + i\pi} \ln \frac{2p^2 - d - 4\mu_0^2}{[(d + 4\mu_0^2)^2 - 4p^2 d] d^{1/4}}$$

$$= (2p^2 - 4\mu_0^2) + \frac{4(p^2 - 4\mu_0^2)}{2 + i\pi} \ln \frac{-2\lambda_0^2 \delta(0) p(-1)^{z/4}(4\mu_0^2)^{1/4}}{\pi(p^2 - 4\mu_0^2)} \tag{3.9b}$$

where p can be p_a, p_b, or p_c according to the Γ_q chosen; then, (3.1) and (3.2) transform to the renormalized 1PI Green's functions reported in the literature for the $\lambda\phi^4$ one-loop diagrams depicted in Fig. 2 (Cheng and Li, 1984).

4. DISCUSSION

In this paper we have presented some ideas and calculations that lead us to propose a general q-regularization scheme and q-renormalization program for quantum field theories, using a noncommutative structure for the internal momentum space in the loop integration.

We use some $\lambda\phi^4$ one-loop graphs as examples to show our procedure. We consider the extension of usual Feynman integrals to Haar weights defined over noncommutative internal momentum space in the loop, but the physically significant external momentum space is the usual one; thus, all the consequences of our noncommutative assumption (for instance, lack of locality) are of no concern from the experimental point of view.

On the other hand, our procedure is strongly dependent on the basis set chosen for C_q^n and its dual. This means that we should handle this property in order to get well-behaved Feynman integrals for all n-loop divergent 1PI diagrams in the $\lambda\phi^4$ theory.

For further work, we consider removing the infinities of some theories constructed on noncommutative geometry.

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REFERENCES

Bacry, H. (1990). The resurrection of a forgotten symmetry: De Broglie's symmetry, in XVIIIth International Colloquium on Group Theoretical Methods in Physics, Moscow, June 1990.

Bullough, R. K., and Timonen, J. (1990). Quantum groups and quantum complete integrability: Theory and experiment, in 19th International Conference on Differential Geometry Methods in Theoretical Physics, Rapallo, Italy.

Cheng, T. P., and Li, L. F. (1984). *Gauge Theory of Elementary Particle Physics*, Clarendon Press, Oxford.

Connes, A. (1988). *Communications in Mathematical Physics*, **117**, 673.

Connes, A. (1990*a*). *Essay on Physics and Non Commutative Geometry. The Interface of Mathematics and Particle Physics*, Clarendon Press, Oxford.

Connes, A. (1990*b*). *Géometrie Non Commutative*, Intereditions, Paris.

Fredenhagen, K. (1981). *Communications in Mathematical Physics*, **79**, 141.

Greenberg, O. W. (1990*a*). *Physical Review Letters*, **64**, 705.

Greenberg, O. W. (1990*b*). Q-mutators and violations of statistics, University of Maryland, Preprint 91-034.

Greenberg, O. W., and Mohapatra, R. N. (1987). *Physical Review Letters* **59**, 2507.

Greenberg, O. W., and Mohapatra, R. N. (1989*a*). *Physical Review D*, **39**, 2032.

Greenberg, O. W., and Mohapatra, R. N. (1989*b*). *Physical Review Letters*, **62**, 712.

Guadagnini, E., Martellini, M., and Mintchev, M. (1990). *Physics Letters B*, **235**, 275.

Majid, S. (1990). *International Journal of Modern Physics A*, **5**, 4689.

Ramberg, E., and Snow, G. A. (1990). *Physics Letters B*, **238**, 438.

Smit, D. J. (1990). *Communications in Mathematical Physics*, **128**, 1.